Mechanics of the escapement driven pendulum

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In principle, the motion of the pendulum can be calculated from its geometry and the forces acting on it. The greatest force by far is gravity, and the lesser ones are stiffness in the suspension, the frictional resistances, and the forces from the escapement that overcome these so as to sustain the motion. The suspension is designed to maintain a geometry that is nearly constant, being a nearly circular path, which makes the force due to gravity nearly proportional to the displacement from equilibrium. The present paper concerns these forces, and how they combine to affect the pendulum motion. It reports work carried out by an engineer, who is not a horologist, but who has been doing research into the history of mechanical clocks and wanted to understand how their actual performance has developed according to technical and scientific advances. The work has attempted to assess the issues that must be addressed, and to show how a combination of experiment and theory is necessary to deal with them.

In practice, the escapement forces are not easy to estimate, either theoretically or experimentally: they are very small, and they depend on friction, play and compliance in the going train, and in the escapement. Frictional resistances are not easy to calculate, but can be measured fairly straightforwardly by looking at the decay in pendulum motion without the escapement, but with the pendulum crutch, connected. The experimental part of the present paper concerns the estimation of these forces, and the measurement of pendulum motion under different conditions of drive and friction. The findings show that suspension, resistance and escapement action have to be separately studied in detail to produce a consistent overall view in which experiment and theory match up.

The pendulum motion can be determined theoretically from the drive and frictional forces by several procedures. The oldest, most versatile, and most easily visualised of these is due to Airy. To see how it works, we can look at the phase diagram, in which the speed of the pendulum is plotted against its displacement (Fig 1) and the speed is scaled so that maximum speed has the same amplitude as maximum displacement. The motion of an ideal pendulum is described by the path of a point going clockwise around a circle in this diagram, as shown. The effect of an additional force is to change the speed, i.e. deflect the path upwards or downwards away from the circular. Thus, for instance, a frictional force that is opposite to the speed will send the path spiralling into the centre by acting downwards in the upper half of the circle and upwards in the lower half. The effect of a force in general can be seen in Fig 1, where it is applied briefly at the point A. The resulting change in speed can be resolved into two components, one that acts along the radius at A, and so changes the amplitude of the motion, and the other that acts along the tangent, and so changes the time of traversal of the whole circle, i.e. the period. Airy’s formulae prescribe the summation of these components throughout the cycle to calculate the resulting motion. The effects of friction, circular error, escapement forces and any other deviation from the ideal can be calculated. This procedure is not exact, because the components have been calculated as if the path remained circular, whereas it is actually made non circular by them. So the deviations have to be small for the formulae to be accurate.

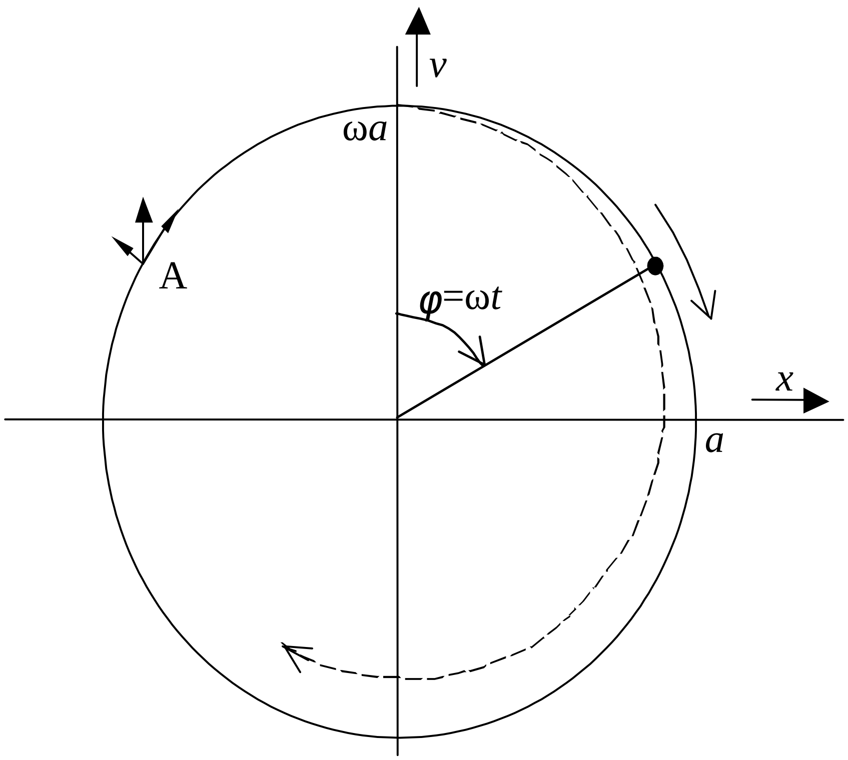


Fig.1. *Phase diagram, showing effect of frictional force (broken line) and impulse at* A

One more recent theoretical work on pendulum motion has sought to improve on this inexactness by considering the path in the phase diagram under a frictional resistance which is proportional to speed[1]. This path is a logarithmic spiral. The effects of a constant force are also neatly calculated by remarking that such a force shifts the centre of the motion along the displacement axis, while it is applied. The effects of anchor and deadbeat escapements are then calculated by the reasonable assumption that the mechanism delivers a succession of constant forces as it progresses through its successive intervals of action. The problem then becomes one of putting together the resulting pieces of logarithmic spiral in the phase diagram to make a closed cycle, which is solved numerically. Deviations from the Airy formulae are detected, but only by looking at cases where the forces are applied with gross asymmetry, and at high levels of friction (a logarithmic decrement of 0.1), neither of which are found in actual clock mechanisms.

Another more recent work[2] uses the first few components of the Fourier series to solve the equation of motion of a nearly ideal pendulum. Unsurprisingly, since it is based on the same mechanical assumption, this produces the same result as Airy, although the author does not refer to the earlier work. The Airy formulae are conveniently reworked in modern notation in another paper[3], which also describes the phase diagram analysis. The paper goes on to demonstrate the use of Green functions in the analysis, because they are particularly suited to the impulsive nature of the force applied by the escapement; the results are similar to Airy's for such a force. Impulsive forces also feature in an analysis of the verge and foliot clock[4], in which frictional losses are ignored.

None of these later papers seem to present an advance on Airy's method, nor useful results on what might happen in real clocks. Useful results on escapement error are reported by Feinstein, as well as alternative numerical methods for the solution of particular cases of non ideal motion[5]. The methods produce virtually identical results, which is reassuring since high accuracy is required for the small effects concerned. The effect of escapement forces is reduced to that of a single impulse, however, and is not directly related to actual mechanisms. The traditional term for the forces applied by the escapement being "impulse" appears to have caused confusion among some authors who have taken it to imply that the forces are impulsive in the dynamical sense, i.e. are of negligibly short duration. To avoid the confusion, the term "drive" will be used in the present paper instead.

Escapement basics

The first task in analysing the effect of the escapement is to assess the forces it applies to the pendulum. Both deadbeat and anchor escapements are designed to deliver constant forces over intervals of the motion by means of a cam action, so we start with that. The standard designs recommend that the pallet moves radial to the escape wheel; in this case the force on the pallet arm due to it being driven forward by the escape tooth is given by

$$F_f = \frac{(1 - \mu s)}{(s + \mu)} P_f$$

where μ is the coefficient of friction, s is the slope of the pallet face with respect to the direction of travel of the tooth, and P_f is the tangential component of force delivered at the contact by the tooth. A similar analysis for the reverse action shows the magnitude of the reverse force is

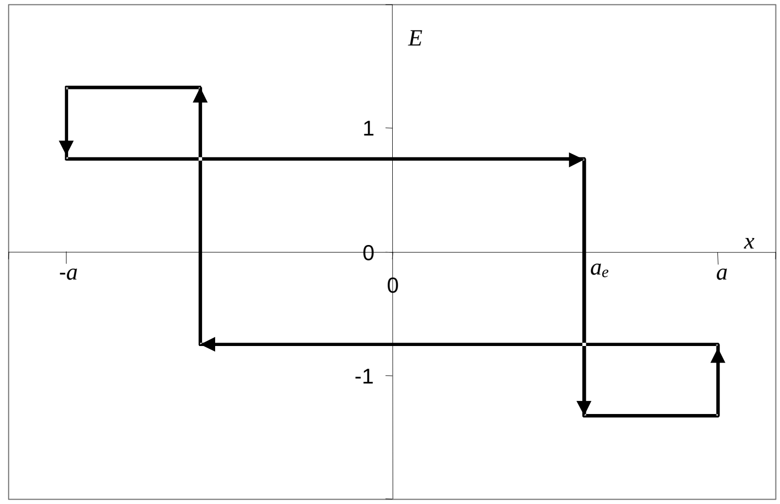
$$F_r = \frac{(1 + \mu s)}{(s - \mu)} P_r$$

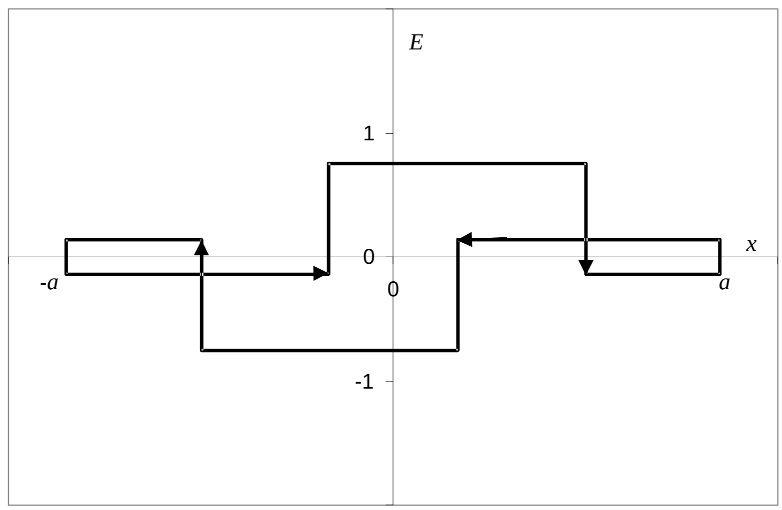
Actually, we are more interested in the fraction of work transmitted, since the escapement is just one part of the power transmission from drive weight to pendulum. This is given for the forward motion by

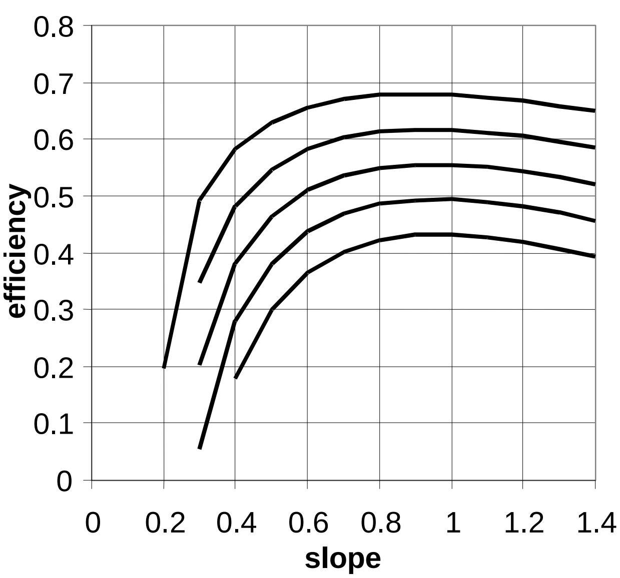
$$E_f = s\frac{F_f}{P_f} = s\frac{(1 - \mu s)}{(s + \mu)}$$

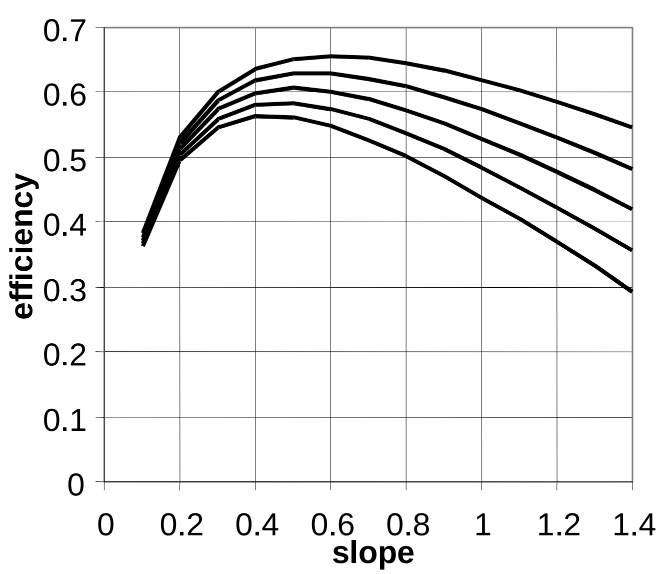
and similarly for the reverse. We can see that if $\mu = 0$ then $E = 1$, as expected. For an anchor escapement, where the direction of the drive is reversed when the direction of sliding is reversed, we might expect $P_r / P_f > 1$ because the forward force is reduced by friction in the train, and the opposite is true for the reverse. It is hard to estimate the size of this ratio in practice, since the reverse motion is very small, and there is play and elasticity in the train to be taken up before the friction acts fully. In the measurements shown below it appears that, for the clock in question, the ratio is approximately unity, so this will be the value used here.

Figs.2 & 3 *Drive loops for an anchor escapement (top) and for a deadbeat.*









Figs.4 & 5 *Efficiency of work transfer by cam action of pallets, for amplitudes from 1.2 to 2 times the escape amplitude, for anchor (left) and deadbeat.*

It is useful to give a plot of *E* against the position of the bob, *x*, to show when the forces act in the motion. These are shown for anchor and deadbeat mechanisms in figs 2 and 3, with arrows to show the direction in which they are executed. The values of *s* used are 0.45 for the deadbeat and 0.9 for the anchor; the value of μ used is 0.15, which is taken from the findings below. To plot *E* in this way is particularly useful, because it is also the force on the bob as a proportion of what it would be with no friction. We use the linear displacement of the bob rather than the usual angular displacement so that the plot relates directly to the mechanics of bob motion. Thus the plot can be used to think about both the action of the escapement, and the accleration of the bob: we shall refer to this plot as the drive loop. The total work delivered to the bob in a cycle is proportional to the area enclosed by the drive loop, remembering that area is positive when the loop is executed clockwise, and negative when anti. Plots of this fraction against *s* are shown in Figs 4 and 5, for μ = 0.15 and various amplitudes. This is the efficiency of the transfer of work by the pallets, and does not include consideration of the lost work in the drop, or elsewhere.

The drive loops also illustrate the basic effect of the escapement on the period of the motion: the anchor loop has an overall negative slope, like the effect of gravity, and so speeds the motion, whereas the deadbeat is opposite. The effect of changes in the escapement force is not so easy to judge, because the amplitude changes as well, but that is dealt with in the next section.

Applying the Airy formulae

To calculate the effect of the drive on the motion we can use the Airy formulae. This is in principle the same as in Rawlings[6], but extended to cover the more realistic drive loops shown above, and generalised suitably for iterative solution. We apply the condition that the net effect of the drive force and the frictional losses are to make the amplitude change zero, so the motion is steady. This reduces to requiring the net work done to be zero; the work going into the bob from the drive is

$$W_i = \frac{W}{4a_e} \oint E\,dx$$

Where a_e is the position of the bob at escape, and *W* is the work done by the escape wheel in a whole cycle (two beats). This integral is the area calculation of the drive loop described above, which is straightforward because the drive loop can be broken down into separate intervals in which the drive force is constant.

The frictional resistances are usually assumed to be proportional to speed, giving a constant decrement when undriven. This may be reasonable for air resistance and other viscous losses, but is not appropriate for the friction in the pendulum crutch, which is more close to a constant resistance. This assumption is supported by the experiments below; using it we have the work lost by the bob is

$$W_o = \pi\beta\omega a^2 + 4ca$$

where β is the proportion of loss to speed, ω is the angular frequency, *a* is the amplitude, and *c* is the constant resistance, all referred to the bob. For a purely proportional loss, *c* = 0 and β = ω / Q. The equation

$$W_o - W_i = 0$$

can then be satisfied by iteration to find *a* if *W* is known, or vice versa.

After this iterative calculation, the fractional gain can be found as

$$r = \frac{W}{\pi\omega^2 a} \oint E(\phi)\sin(\phi)d\phi$$

Using the previous assumption, that the drive action can be broken down into intervals in which *E* takes a constant value, let E_i be the value of *E* over interval *i*, and the above integral becomes the sum of straightforward integrals:

$$r = \frac{W}{\pi\omega^2 a} \sum_i E_i \int \sin(\phi)d\phi$$

$$= \frac{W}{\pi\omega^2 a} \sum_i E_i \left[\cos(\phi) \right]_{\phi_i}^{\phi_{i+1}}$$

$$= \frac{W}{\pi\omega^2 a} \sum_i E_i \left(\sqrt{1 - \left(\frac{x_{i+1}}{a} \right)^2} - \sqrt{1 - \left(\frac{x_i}{a} \right)^2} \right)$$

which is readily evaluated from the vertices of the drive plot.

Relating the theory to what might realistically happen to change the rate of a clock.

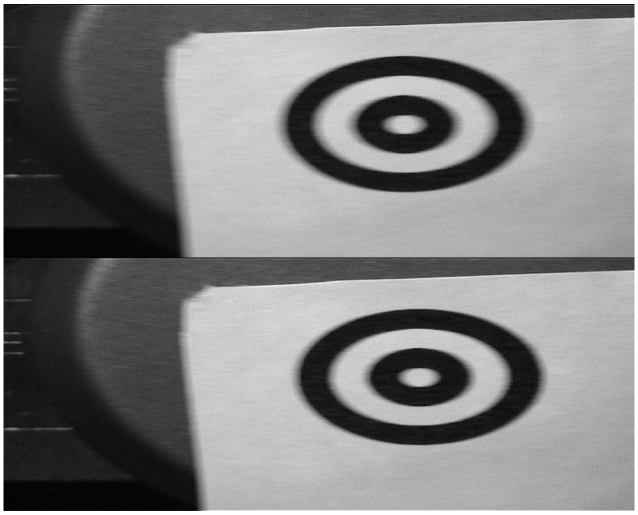
The frictional losses can be expected to change with time after lubrication. The escapement friction coefficient, μ, and the resistance of the crutch, *c*, will increase. The value of *W* will also change, being decreased by losses in the gear train and escape wheel bearings. The results of the calculations for various changes in these three parameters have been explored by writing a program that takes text input from a spreadsheet to define the parameter values, and the ranges over which they are to change, and presents the results in a format that can be pasted into a spreadsheet for display. Plots of results from this program are shown below, where they are compared with results from experiments.

Experiments have also been carried out with a mantel clock, driven directly by a spring in a barrel and beating three times a second. The clock was the Westminster chime model of the Enfield Clock Co., made c.1933, in the “Napoleon hat” style. The pivots and pallets were dismantled, cleaned and minimally lubricated with household oil, and manually aligned on reassembly. Some wear was observed on the pallets, but they had enough depth to be aligned so that the escapement bore on unworn areas. The clock was tested immediately after cleaning and lubrication, then run for three months, and retested. Small weights were hung on the hour hand to produce changes in the drive force. Measurements were taken using a video of a marker attached to the pendulum bob, so that period and amplitude could be monitored together. The system for taking these measurements, and assessing the results is described in the next section.

Marker tracking and free pendulum motion

An illustrative frame from a video sequence is shown in Fig. 6. The marker is a circular black and white pattern, the position of which is located by detecting the edges of each ring. The marker is designed so that there is as much edge as possible, so more than one ring is used. The thickness of each ring has to be sufficient for it to be clearly measurable despite the motion blur, so for the case of this pendulum that means two rings, each providing an inner and an outer edge. The video camera is a consumer mini DV camera, and takes interlaced frames of 720x576 pixels at 25 frames per second. In each frame the even lines are scanned, then 1/50th of a second later the odd lines are scanned, so the frame shown is the composite of these two scans. The scans can be separated in software, and displayed as two separate half frames, each with half the vertical scale of the original, and much less blur: this is shown in Fig.7. These pictures are then measured separately, and the results scaled back to the original, so as to produce measurements at 50fps.

Figs. 6 and 7. *Images of pendulum bearing a marker, as captured (left) and with the interlaces separated.*



The software requires the user to indicate a starting point inside the outer ring in one half frame, and then searches from that point for the outer edge of that ring, and tracks the edge around. An ellipse is then fitted to the resulting edge pixel coordinates, so that the extent and centre of the edge profile can be determined. The inner edge is then found and fitted, and from these data the inner ring is located and its edges tracked. The mean centre coordinates of the fits and the dimensions of the outermost edge fit are stored. The corresponding starting point in the other half frame is calculated, and its marker image is similarly processed. The velocity is calculated, and used to predict the starting point for the next pair of half frames; in this manner the entire sequence of frames are processed.

The output then consists of a sequence of frame numbers and pixel coordinates of the centre of the marker in each frame. To find the period, amplitude and mean of the motion, a sine curve is fitted to a subset of these coordinates at each frame in the sequence. The best size of the subset to remove numerical artifacts appears to be so that it covers 1.5 times the period, i.e. 50 frames. The periods are then averaged over 100 frames, to remove residual artefacts. Fig 8 shows the resulting data for the pendulum entirely free, at three different amplitudes, as a plot of gain in seconds per day against number of cycles. The result at the lowest amplitude shows an erratic variation of about ±0.7s, which can be taken as an assessment of overall error in the method, as far as timing is concerned. Some of this variation may not come from inaccuracy in the measurements: the data at higher amplitudes has larger variations, which are presumably due to genuine mechanical effects of the suspension. Some of the variation at the lowest amplitude may also be genuine, and so the underlying random error may be considerably less than the overall figure quoted above. Experiments in which two smaller targets are attached to the bob, and their separation measured in the same way, indicate that the error in separation measurement is about ±0.1pixel, i.e. ± 5μm at the resolution used here. The error in positional measurement of the single target would be about half of that value.

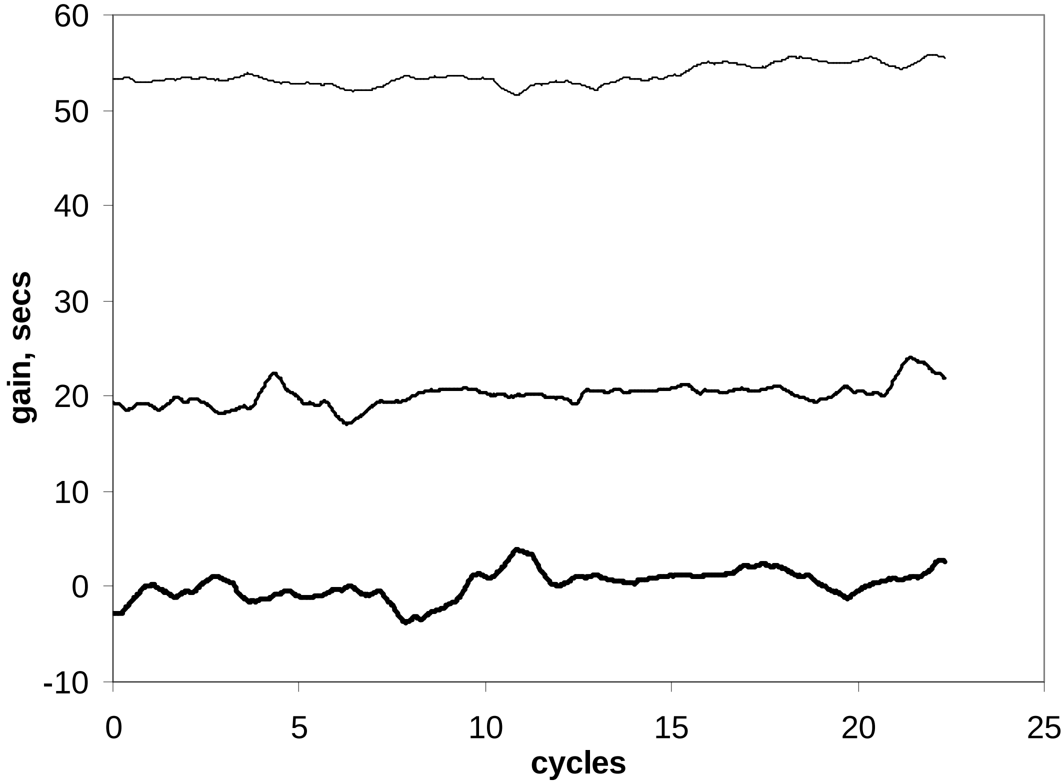


Fig.8 *Relative gain versus cycles of free pendulum motion at amplitudes of 6.4mm, 5.2mm and 3.6mm, on a pendulum of theoretical length 110mm.*

An important phenomenon emerged from this data on a free pendulum when it was compared to the calculated circular error: the gain is about 3.2 times the calculated error. The cause of this has been traced to the suspension spring. The pendulum bob was detached, and the motion of the upper part of the shaft (a plated steel pressing about half the length of the pendulum) on its own was measured. A further measurement was made with a mass of 2.27 gm attached to this shaft at a radius of 55mm. In this way the returning moment of the spring was made similar to that of gravity, and the effect of amplitude on gain was greatly increased, and thus could be estimated. The rate error of the spring alone was about $7.3\alpha^2$, where α is the angular amplitude. The data also enabled an estimate of the moment of the spring: the spring was equivalent to a weight of 2.5gms on the bob, as compared with the bob weight of 120gms. According to these calculations, we would expect the effect of the spring error on the pendulum complete with bob to be about 2.4 times that of the circular error due to gravity, in good agreement with the data obtained by direct measurement.

Measuring the running pendulum

Motion measurements were made with the pendulum crutch connected, but the escapement disconnected. A logarithmically decrementing sine curve was fitted to these, at each of three amplitudes, so as to provide an estimate of the decrement. The decrement varied with amplitude, indicating a nonproportional damping. The decrement for proportional damping is

$$D_p = \pi\beta/\omega$$

and for a constant resisting force is

$$D_c = \frac{4c}{M\omega^2 a}$$

If *D* is the total decrement then

$$aD = a\pi\beta/\omega + 4c/M\omega^2$$

so that if this expression describes the losses, a graph of *aD* against *a* should be linear. This is shown in Fig.9, with a linear fit. We note that for proportional damping, this graph should pass through the origin: clearly the constant resistance is substantial. From the parameters of the fit, the magnitude of the proportional and constant resistances were found: these are used in the comparisons with theory that are shown below. The proportional element of the damping corresponds to a Q of about 900, which seems reasonable for this movement. The constant resistance equates to an acceleration of 0.78mms^{-2} at the bob. This can be compared with what would arise from friction (coefficient 0.15) at the arbor pivots (diameter 0.75mm) under the weight of the pallet/crutch assembly (11gms), which would give 0.47mms^{-2} at the bob. The measured value is high, but the calculated one is ideal, and assumes no play in the pivots: a play of 20μm alone would explain the discrepancy.

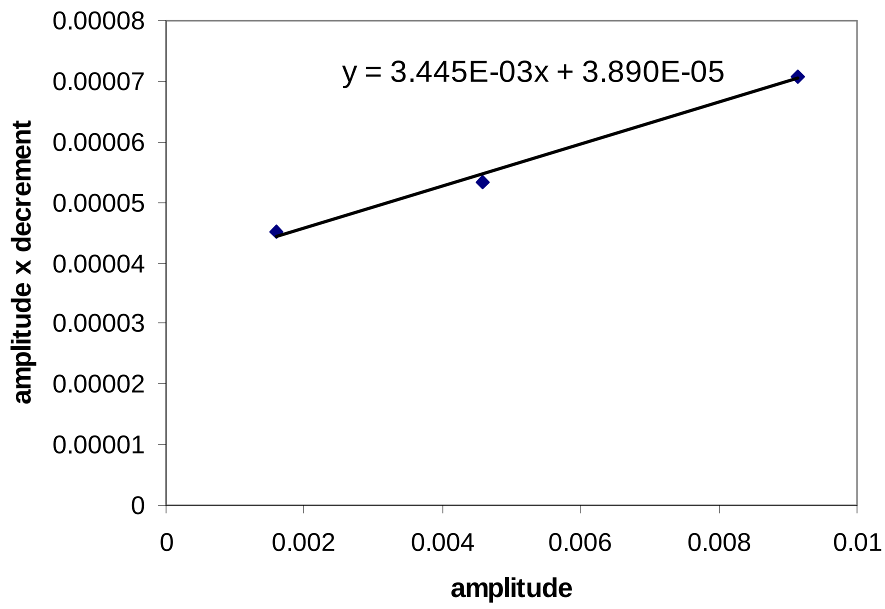


Fig. 9 *Plot to evaluate constant and proportional elements in the pendulum friction (see text)*

Motion measurements were also made with the clock in operation. Fig 10 shows a plot over about one minute of the variations of the mean postion, amplitude and period of the motion. We note that there are cyclic variations in period and amplitude, a slow one having a period of about 45 pendulum periods, and a fast one having a period about seven times faster. Since there are 45 teeth on the escape wheel, and seven teeth on its pinion, these would appear to come from variations in the drive geometry.

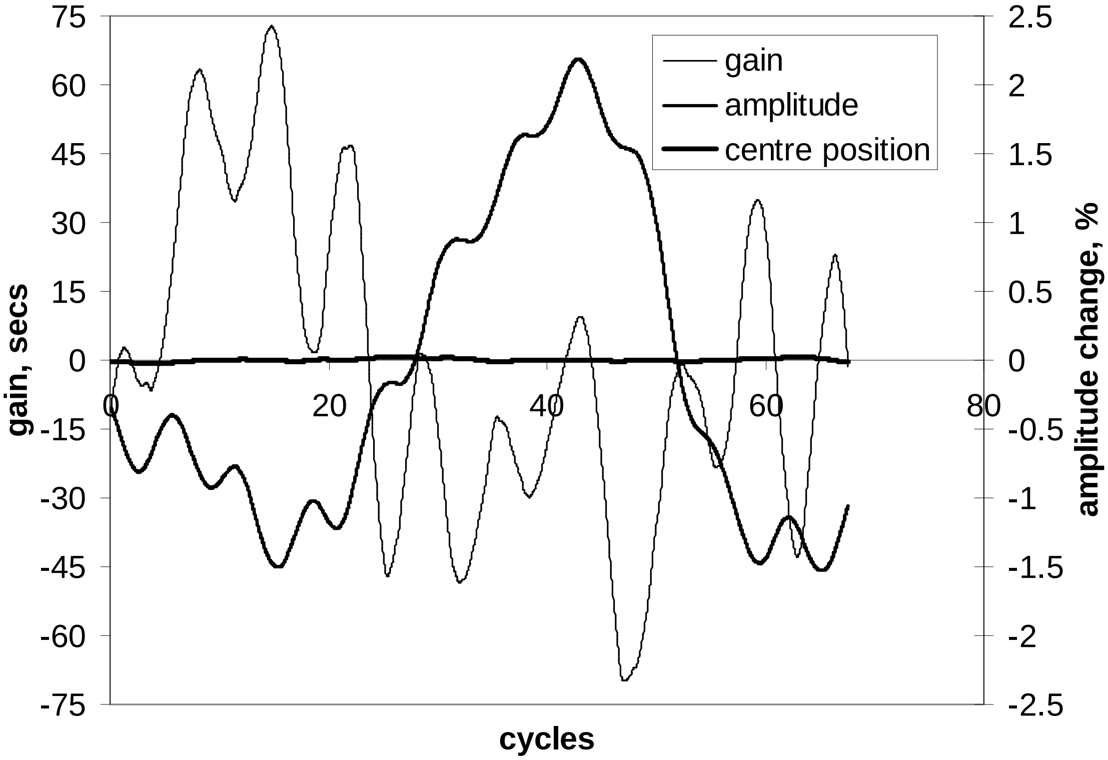


Fig. 10 *Variation of measured parameters when running*

Effects of drive force variation and lubrication

The measurements cover three sets of conditions: the lubricated state with various reduced drive forces; the variation between just lubricated and run in states at full drive force; and the run in state with various drive forces.

Motion measurements were made with weights hung on the hour hand to oppose the drive; it was found that eight grammes was sufficient to make the operation of the clock barely sustainable. Motion measurements were made with weights of 2, 4, 6 and 8 gm applied, in an ascending and descending sequence. The results were compared with that obtained from the numerical estimates based on measured drive geometry and the Airy equations, as described above. It was found that with the friction coefficient on the pallets set to 0.15, a reasonable agreement with the measured values was obtained, as shown in Fig.11.

Similar measurements were made after the clock had been running for three months, by which time it was losing about 100 secs per day. At this stage, it was found that running was just unsustainable with 6gm applied. Various trials with setting the parameters indicated that an increase in the constant frictional resistance of the crutch by 73%, a reduction in the drive by 33% and an unchanged pallet friction produced gain and amplitude results from the numerical computation that matched the measurements. Results of the measurements and computation are also shown in fig.11.

The combined effect of circular error and spring nonlinearity, as determined in the free pendulum experiments, was included in the above calculations. For this clock, the escapement error is about 25 times as large as the circular error, or eight times larger than the combined effect of circular error and spring nonlinearity.

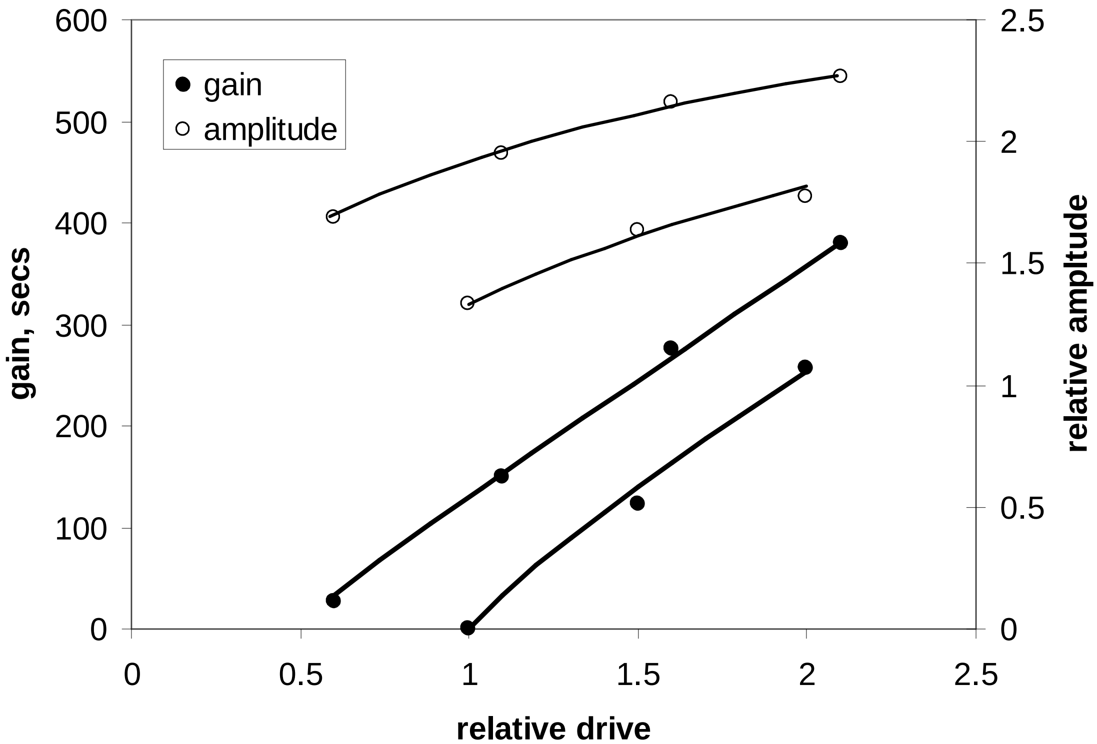


Fig. 11 Gain, and amplitude relative to the escape amplitude, against drive for calculated (lines) and measured (points) values, after lubrication and after three months running.

Acceleration of the pendulum

Since the pendulum is driven, we might hope to detect the accelerations produced by the drive by analysing its motion. The position data was numerically differentiated twice with respect to time to obtain the accleration, and the calculated acceleration of the free pendulum was subtracted from it, to leave the drive element. The result is shown in Fig.12 for a few cycles of the motion, in the same manner as the theoretical forces of Fig.4 are displayed. Also included in the figure is the acceleration obtained from the theoretical model: that is from the forces of fig.4 combined with the resistive forces obtained from the loss estimate made above, divided by the mass of the bob. The net area of this total acceleration loop is zero, since the motion is steady. Fig 12 also shows the accelerations observed at some of the reduced drive forces.

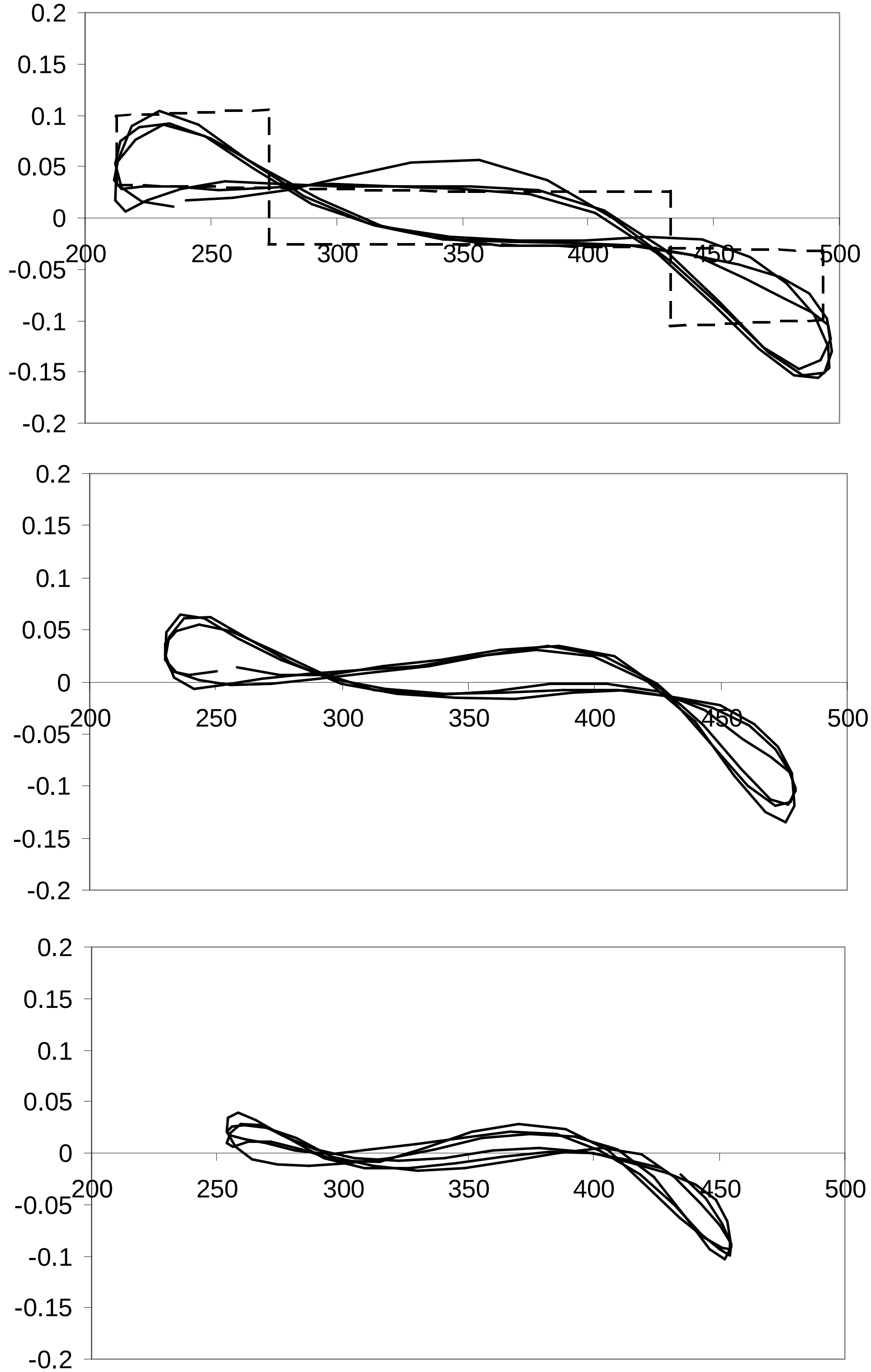


Fig. 12 *Acceleration of bob at three levels of drive after running in, from full drive at top to minimal at bottom. The units are native to the measurement process, and the broken line shows the calculated acceleration.*

Discussion

It is essential in making calculations of effects of changes in conditions on the escapement error to find their effect on amplitude as well. The software written to do this is straightforward, and can be made freely available, but it is designed for the author's personal use, so some development would be worthwhile if there was interest from others in using it. An important feature of the present work is that it has been possible to match measured performance with the escapement theory, with a minimum of assumptions. In carrying this out, it was observed that having the amplitude measurements as well as the rate measurements was vital in determining the parameters. For example, my initial guess at pallet friction coefficient being 0.3 led to a calculated variation of amplitude that was much smaller than that observed. The value of 0.15 which produced a good match turned out to be more reasonable according to expert tribological opinion.

The calculation of the efficiency of operation of the pallets is straightforward, but the formulae and plots are presented here because they are not in agreement with some that have been made available on the web[7]. We note that the maximum efficiency of each escapement is found when the drive slope is near to the recommended standard value, but the exact location of the maximum depends on the coefficient of friction. Trials with varying the friction coefficient indicate that the calculated maximum efficiency is at the standard slope when the coefficient is slightly higher than the value found here. This is reasonable, since attaining maximum efficiency to keep the clock going is most important when lubrication has deteriorated.

The video technique used in this paper is low cost (particularly if you already have a video camera) and remarkably accurate. Positional errors are about 3um, and at 50 readings per second this makes detailed study of pendulum motion possible. Rate can be measured to better than one second per day by only two seconds of observation: more extended periods will produce lower errors, by averaging, but of course since rate varies when running the meaning of such an average is not clear. The accuracy of the camera clock would also have to be verified, since it is unspecified, and not necessarily stable over extended periods. The software for making the measurements could also be made freely available, with the same caveat as above.

The losses in the pendulum crutch for the clock tested here are about twice as large as the proportional losses which presumably come from air resistance, and damping in the suspension mounting. An estimate of escapement error based on the Q value obtained from the free pendulum would be far under the actual value. With the video measurement accurate and detailed estimates of loss can be made under a range of conditions for the undriven pendulum.

The contribution of the suspension spring to the variation of rate with amplitude is over twice that from the circular error. The spring contributes about 2% of the returning force, so we may adopt this as a design guide for the suspension: that for every one percent of the returning force that is supplied by the spring, we may expect a 100% contribution to the circular error. The present spring is in two strips each 2mm wide and has a length of 5mm and thickness of about 0.04mm. The spring has about 20 times the stress from bending as it has from the weight of the bob, so it is thicker than needed from a strength point of view. A more flexible spring would provide less error, but also less well defined geometry of pendulum motion, which may lead to other problems.

The direct measurement of acceleration due to the escapement, and its match to the acceleration deduced theoretically from fitting the response of the pendulum to variations in drive force is particularly pleasing. The measured acclerations are not sharp square plots like the theoretical ones because each numerical differentiation uses four measured points, so the values of these points are averaged, and any sharp corners would be blurred as a result. The measured acclerations show a considerable asymmetry, most marked in the difference between the extreme left and the extreme right of the plots, where the direction of swing is reversing. At the left, the acceleration dips sharply, down to a value slightly below the steady left to right level. At the right, the magnitude of the accleration remains greater than the steady right to left level, before settling down to it. The latter phenomenon implies that some of the energy of the drive is being stored: this may be happening because of play in the escape wheel bearings. At the right of the plot the escape wheel is being lifted by the pallets, whereas at the left it is being pushed downwards. The asymmetric form of the plot is consistent with the weight of the escape wheel delivering the extra acceleration at the right as it descends, but digging in at the left and increasing friction, and so taking away from the delivered force at the left.

It would be of interest to extend this work to higher quality clock mechanisms, with longer and heavier pendulums. In these, resistance from a plain bearing crutch would have a similar moment, and so produce much smaller acceleration. The contribution of the suspension spring may also be smaller. The acceleration due to the escapement would be much smaller and less easily measurable, but the speed of operation is also slower, so more data and higher accuracy will be available.

Calculated errors for other cases

Since we have verified the escapement error calculations, obtained values for frictional losses, and estimates of how they might change with use, we may now calculate the effect of such changes on other mechanisms. The changes correspond to those found in the tests reported above, which are not necessarily typical of other running conditions. A pendulum of period 2s, escapement amplitude 20mm (about 1°) and bob mass 1Kg was examined. The table below shows results as follows:

Case A

Long case clock with anchor escapement, subject to similar changes in friction to that found in the mantel clock, with a similar relative amplitude. A value of Q = 2000 has been assumed, and the crutch friction adjusted so that on a drive power of about 20 lb.in/day it produces adequate amplitude. The initial condition corresponds to the commonly found excessive amplitude and drive power input. Escapement error is about ten times circular error.

Case B

As above, but with deadbeat escapement fitted, to see the effect of just a change in escapement. Escapement error is reduced, and circular error increased: as it happens, they almost balance, but, as Airy commented, this is a coincidence that cannot be securely obtained.

Case C

A deadbeat in a more likely context: a regulator, Q = 7000, with less variation in drive and friction, and closer running. The escapement error is not much reduced: this is because the smaller amplitude makes the error larger. This is illustrated in the next case.

Case D

As case C, but larger amplitude.

	A Longcase		B Deadbeat		C Regulator		D Regulator	
	initial	final	initial	final	initial	final	initial	final
rel. drive	1.5		1.5		1.3		1.3	
pallet friction μ	.15	.15	.15	.15	.15	.15	.15	.15
prop loss β / M s^{-1}	.0015	.0015	.0015	.0015	.00045	.00045	.00045	.00045
con loss c / M ms^{-2}	.000075	.00014	.000075	.00014	.000075	.0001	.000075	.0001
rel. ampl. a / a_e	2.04924	1.6	2.26807	1.6	1.76549	1.3	1.99869	1.5
work/cycle W_i μJ	37.1631	33.0797	44.0707	33.0797	16.1304	13.4023	19.0889	15.9972
esc. err.		-32.9588		-6.88307		-5.73703		-3.90513
circ. err.		3.5842		5.64975		3.11983		3.81461
efficiency	0.4158	0.5552	0.5372	0.6048	0.5881	0.63525	0.5645	0.615

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